Two-frequency mutual coherence function and pulse propagation in random media

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(Received 23 July 2001; published 10 April 2002)

In this work an analysis of transient wave propagation in forward scattering random media is presented. The analysis is based on evaluation of the two-frequency mutual coherence function, which is an important quantity in itself since it provides a measure of the coherence bandwidth. The coherence function is calculated by using the path integral technique; specifically, by resorting to a cumulant expansion of the path integral. In contrast to the formulas available in the literature, the solution obtained is not limited by the strength of disorder and applies equally well to both dispersive and nondispersive media, with arbitrary spectra of inhomogeneities. For the regime of weak scattering (or relatively short propagation distances) the first cumulant gives an excellent approximation coinciding with the results obtained earlier in a particular case of the Kolmogorov turbulence by solving the corresponding differential equation numerically. In the regime of strong scattering (long distances), which to our knowledge has not been covered previously, our solution demonstrates a different type of scaling dependence. It is shown that, even for power spectra with fractal behavior in a wide range of spatial frequencies, the coherence function is very sensitive to fine details of the spectrum at both small and large spatial scales. Using the cumulant expansion, the temporal moments of the pulsed wave propagating in a random medium are also considered. It is found that the temporal moments of the pulse are determined exactly by accounting for a corresponding number of the cumulants. In particular, the average time delay of the pulse is determined by the first cumulant, and the pulse width is obtained by accounting for the first two cumulants. Although the consideration of the problem is based on the model of a continuous medium, the results are also applicable to wave propagation in media containing discrete particles scattering predominantly in the forward direction.

DOI: 10.1103/PhysRevE.65.046617 PACS number(s): 42.25.−p, 43.20.+g

I. INTRODUCTION

In this work we study a two-frequency coherence function and temporal evolution of transient (pulsed) waves propagating along directed paths in random media. Along with the generic significance of the subject in the physics of disordered systems where the propagation of both quantum mechanical and classical wave packets is of interest (see, e.g., Refs. [1–3]), there is a variety of applications dealing with ultrawideband signals transmitted in a complex environment. In particular, high data rate communication systems at radio and optical frequencies may be influenced by pulse spreading due to the scattering by turbulent inhomogeneities and hydrometeors in the troposphere, and by electronic concentration fluctuations in the ionosphere. Moreover, in the latter case, the random scattering is superimposed on the background effect of dispersive spreading. In contrast to the pulse spreading in a homogeneous temporally dispersive medium, which may be compensated for in the receiver, the same effect caused by spatial dispersion of random media leads to an irreversible degradation of the transmitted signal. Other applications, also dealing with randomly scattered short pulses, include radars and other remote sensing schemes, underwater acoustics, and the interpretation of signals emitted by extraterrestrial radio sources such as pulsars [1,4,5].

The basic phenomena of the transient propagation process can be studied in the framework of a space-time ray approach according to which the wave process is considered in terms of wave packets moving along complex space-time rays, permitting the analysis of wave fields with exponential temporal and spatial amplitude variations [6]. However, the ray technique is limited when studying fine dispersion and diffraction structures of the wave field, especially in random media where many multiple-scattered waves come to the observation point. In such a case, we should resort to a full stochastic wave equation describing the wave field varying in both space and time. Another option is to solve the reduced form of the wave equation, written for the time-harmonic field, and to return to the time-dependent field by using an appropriate Fourier transformation. The latter method is adopted here to study the transient wave propagation in random media. It is worth noting that this indirect procedure is the only possibility in many situations dealing with dispersive media, for which the wave equation in the time domain is unknown, and the properties of the medium are described by a dispersion relation presented in the frequency domain. Within the framework of this formalism, the complete information about transient propagation requires a solution for the statistical moments of the wave field at different frequencies and at different positions [7]. In particular, to evaluate the average pulse shape one has to know the two-frequency mutual coherence function, which is also an important quantity in itself since it provides a measure of the coherence bandwidth [1].

As is known, the propagation of directed waves can be described with good accuracy by a parabolic-type wave equation for the complex amplitude. In the Markov approximation, i.e., when the inhomogeneities of the medium are supposed to be δ correlated along the direction of wave propagation, the two-frequency mutual coherence function also satisfies an equation of the same type that was obtained in [8]. Some attempts were made to analyze this equation by using either a rather cumbersome eigenfunction expansion developed for a power-law (fractal) medium [9] or a two-scale procedure which, in principle, is capable of describing random media with any given statistics [10]. The latter method, however, leads to a multiple integral of a highly oscillating function which requires considerable numerical effort to be applied to complete the analysis of the problem. In both cases the control of the accuracy is an additional difficulty.

The only known analytic solution of the equation for the coherence function is based on the approximation of the transverse structure function by a quadratic form [11]. In the framework of the path integral approach adopted in our work, this result is not surprising because in this case there is a soluble quadratic type Lagrangian, and the path integration can be performed exactly [12]. The quadratic approximation, however, corresponds to accounting for random tilts of the wave front, while the small-scale perturbations causing the pulse spread are completely neglected. In this case, the broadening of the ensemble-averaged pulse is due entirely to the fluctuations in arrival time of the pulse, which remains unperturbed in each particular realization [13]. Obviously, such a model cannot be adequate for the description of signals transmitted through random media, and a more acceptable technique for tackling the problem should be developed. Although a quarter of a century has passed from the time the solution based on a quadratic approximation was obtained, no serious progress in this area seems to have been achieved.

An important exception that must be mentioned is [14] in which the temporal evolution of pulsed signals was studied, and the problem of finding the mutual coherence function itself was avoided. Indeed, it was realized that for calculation of temporal moments of the pulse (such as mean arrival time and pulse width) it is sufficient to evaluate the derivatives of the coherence function for zero frequency separation. Obviously, the description of transient signals by a number of temporal moments is sufficient only for pulses of simple shapes, like, for example, quasimonochromatic wave packets with Gaussian envelopes. Propagation of transients with complex spectral content, e.g., frequency modulated or ultrawideband, especially in dispersive media, may cause formation of signals of very intricate form, and even disintegration of the initial signal into pulse trains. In this situation we should resort to a full description based on the mutual coherence function, not its derivatives. Moreover, the coherence function itself is needed in many cases since it is an easily measurable quantity and may be used in inverse problems aimed at characterizing the properties of the scattering medium.

In this paper we show that the two-frequency mutual coherence function can be evaluated with good accuracy by using a path integral technique supplemented by the Markov approximation, for any type of fluctuating medium (including both dispersive and nondispersive media) and for any type of the disorder (single correlation scale or fractal media), as well as for any strength of randomness. Although the model of a continuous medium is considered here, the results obtained are also applicable to wave propagation in media containing discrete particles that scatter predominantly in the forward direction. This possibility arises from the well-known fact that, under certain conditions, the equation for the mutual coherence function can be reshaped into a two-frequency radiative transfer equation [1,8].

It should be mentioned that this problem can be considered even in a wider physical framework including other formulations, which are similar in form. In fact, the parabolic wave equation used to model the propagation of directed classical waves coincides with the nonstationary Schrödinger equation that describes the motion of a quantum particle in a random time-dependent potential [15–18]. The analog of time for classical waves is the range coordinate, and the random potential corresponds to the spatial fluctuations of the refractive index. Moreover, the imaginary time version of the Schrödinger equation describes the problem of directed polymers in a random medium [19]. When, in addition, the potential is also imaginary, then the model is relevant to quantum tunneling of a strongly localized electron under a random barrier [20].

The outline of the paper is as follows. In Sec. II, the general relation between the mean shape of the transient wave and the two-frequency mutual coherence function is presented. Then, in Sec. III, the path integral approach adopted in this work and the cumulant technique used for the evaluation of the path integrals are described. In Sec. IV, the path integral technique is applied to the calculation of the coherence function for a dispersive medium with a homogeneous background. In Sec. V, the results obtained are analyzed and exemplified by considering the particular case of a generalized Kolmogorov turbulence. In Sec. VI the relations between temporal characteristics of the pulsed waves and the two-frequency mutual coherence function are introduced. For the model of a narrowband signal propagating in a nondispersive medium, both the mean arrival time and the pulse width are calculated. The final section contains a summary and some concluding remarks.

II. TEMPORAL EVOLUTION OF TRANSIENT WAVES

To study the temporal evolution of transient waves in random media, we consider the time-dependent field $\psi(\mathbf{r},z,t)$, which can be presented as a superposition of time-harmonic waves $U_\omega(\mathbf{r},z)$ satisfying the reduced Helmholtz equation. Here the z axis is chosen along the direction of wave propagation, and $\mathbf{r}$ is the coordinate in the transverse plane. In the paraxial approximation the field $U_\omega(\mathbf{r},z)$ is presented as

$$U_\omega(\mathbf{r},z)=\exp[ik(\omega)z]u_\omega(\mathbf{r},z), \tag{2.1}$$

where the wave number $k(\omega)$ describes the spectral properties of the unperturbed background medium, and the complex amplitude $u_\omega(\mathbf{r},z)$ is governed by the standard parabolic wave equation containing a random perturbation of the scattering potential [1]. Thus, the time-dependent field has the form

$$\psi(\mathbf{r},z,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\,\exp[-i\omega t+ik(\omega)z]\times u_{\omega}(\mathbf{r},z)S(\omega), \tag{2.2}$$

where $S(\omega)$ is the spectrum of the excited pulse.

When dealing with wave propagation in random media, we are interested in finding the statistical moments of the signal intensity,

$$\langle I^n(\mathbf{r},z,t)\rangle\equiv\langle|\psi(\mathbf{r},z,t)|^{2n}\rangle,\quad n=1,2,\ldots \tag{2.3}$$

(the angular brackets denote ensemble averaging). In particular, the first ($n=1$) intensity moment characterizing a mean shape of the pulse is of major interest [1]. Using Eq. (2.2), we obtain for the mean intensity

$$\langle I(\mathbf{r},z,t)\rangle=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}d\omega\int_{-\infty}^{\infty}d\Omega\,\exp(-i\Omega t)\times\Phi(z,\omega,\Omega)\Gamma(\mathbf{r},z,\omega,\Omega), \tag{2.4}$$

where

$$\Phi(z,\omega,\Omega)=\exp[ik(\omega+\Omega/2)z-ik(\omega-\Omega/2)z]\times S(\omega+\Omega/2)S^*(\omega-\Omega/2) \tag{2.5}$$

is the bilinear spectrum of the transient plane wave propagating in a homogeneous medium and measured at a distance z from the source, and the correlator

$$\Gamma(\mathbf{r},z,\omega,\Omega)=\langle u_{\omega+\Omega/2}(\mathbf{r},z)u^*_{\omega-\Omega/2}(\mathbf{r},z)\rangle \tag{2.6}$$

is the mutual two-frequency coherence function. According to Eq. (2.4), the mean shape of a wave packet is determined by two factors. The first, the bilinear spectrum $\Phi(z,\omega,\Omega)$, accounts for the distortion of a transient plane wave propagating in a dispersive medium. The study of this effect is a classic topic of textbooks treating the propagation of pulsed waves in dispersive media. The second factor, the two-frequency mutual coherence function $\Gamma(\mathbf{r},z,\omega,\Omega)$, describes the scattering of the wave on random inhomogeneities, an effect depending essentially on the frequency, due to both the temporal and spatial dispersion of the medium. In random media, the second effect can dominate and provide the main contribution to the pulse spread.

The important question that should be discussed and has been just mentioned casually in Sec. I is how the shape of the pulse averaged over the ensemble is related to the shape of the pulse in an arbitrarily taken realization. In random media with only one correlation scale (such as atmospheric hydrometeors) there is essentially no difference between these two values, i.e., the pulse shape obtained by performing ensemble averaging coincides with that observed in each particular realization. In media with fractal spectra (such as Kolmogorov turbulence) there are two mechanisms that cause broadening of the ensemble-averaged pulse: the pulse may be spread in time as a result of multiple scattering for each realization of the ensemble, and it can also spread in time as a result of averaging over fluctuations in the arrival time in different realizations [13]. However, the latter effect is significant under the conditions of extremely weak scattering (in the regime of weak intensity fluctuations), and is described mainly by a separable factor which will be deliberately ignored in our analysis.

III. PATH INTEGRAL APPROACH

We start with the parabolic equation for the Green's function,

$$2ik(\omega)\partial_z g_{\omega}+\nabla^2_{\mathbf{r}}g_{\omega}+k^2(\omega)\tilde{\epsilon}(\mathbf{r},z,\omega)g_{\omega}(\mathbf{r},z|\mathbf{r}_0,z_0)=0, \tag{3.1a}$$

supplemented by the initial condition

$$g_{\omega}(\mathbf{r},z_0|\mathbf{r}_0,z_0)=\delta(\mathbf{r}-\mathbf{r}_0). \tag{3.1b}$$

It is assumed that the fluctuating parameter (permittivity, or scattering potential) $\tilde{\epsilon}(\mathbf{r},z,\omega)$ is a Gaussian random field with zero mean value, $\langle\tilde{\epsilon}(\mathbf{r},z,\omega)\rangle=0$, i.e., the medium is characterized by a homogeneous background.

The parabolic equation (3.1) coincides with the nonstationary Schrödinger equation that describes the motion of a quantum particle in a random time-dependent potential. Using this analogy, the solution of the equation can be presented in the Feynman path integral form:

$$g_{\omega}(\mathbf{r},z|\mathbf{r}_0,z_0)=\int_{\mathbf{r}(z_0)=r_0}^{\mathbf{r}(z)=\mathbf{r}}D\mathbf{r}(\zeta)\exp\left(i\frac{k(\omega)}{2}\int_{z_0}^{z}d\zeta\{\dot{\mathbf{r}}^2(\zeta)+\tilde{\epsilon}[\mathbf{r}(\zeta),\zeta,\omega]\}\right), \tag{3.2}$$

where the integration $\int D\mathbf{r}(\zeta)$ in the continuum of possible trajectories is interpreted as the sum of contributions of arbitrary paths over which a wave propagates from point $\mathbf{r}_0$ at the "moment" z_0 to point $\mathbf{r}$ at z, and the expression in the exponent may be considered as an "action functional" which is related to the phase accumulated along the corresponding path [21,22].

By changing the "integration variable"

$$\mathbf{r}(\zeta)=\mathbf{r}_0+\int_{z_0}^{\zeta}d\zeta\,\mathbf{v}(\zeta), \tag{3.3}$$

we turn to the so-called velocity representation, which can simplify analytical transformations of the path integral [22]. In this case $\mathbf{v}(\zeta)=\dot{\mathbf{r}}(\zeta)$, and in two dimensions the new integration variable can be related to some velocity that explains the name of this representation. According to Eq. (3.3), the condition $\mathbf{r}(z_0)=\mathbf{r}_0$ is satisfied automatically. The second end restriction for each trajectory can be specified by a δ function in the integrand. This leads to

$$g_{\omega}(\mathbf{r},z|\mathbf{r}_0,z_0)=\int D\mathbf{v}(\zeta)\,\delta\left(\mathbf{r}-\mathbf{r}_0-\int_{z_0}^{z}d\zeta\,\mathbf{v}(\zeta)\right)\times\exp\left\{i\frac{k(\omega)}{2}\int_{z_0}^{z}d\zeta\left[\mathbf{v}^2(\zeta)+\tilde{\epsilon}\left(\mathbf{r}_0+\int_{z_0}^{\zeta}d\zeta\,\mathbf{v}(\zeta),\zeta,\omega\right)\right]\right\}. \quad (3.4)$$

Using the path integral representation, in the form of either Eq. (3.2) or (3.4), we can write down any statistical moment of the wave field and perform an ensemble averaging provided the statistics of the medium is known. The resulting expression for an Nth-order statistical moment of the field (in our case this is the second-order two-frequency mutual coherence function) has the generalized form

$$\Gamma=\int D\boldsymbol{\rho}(\zeta)\exp\{-X[\boldsymbol{\rho}(\zeta)]\}, \quad (3.5)$$

where $\boldsymbol{\rho}(\zeta)$ is a set of N Feynman paths, either in a regular coordinate space or in the velocity representation, i.e., $\boldsymbol{\rho}(\zeta)\equiv\{\mathbf{v}_1(\zeta),\ldots,\mathbf{v}_N(\zeta)\}$, and it is assumed also that the end restrictions and the free-space parts of the action functionals are included in $D\boldsymbol{\rho}(\zeta)$. The functional $X[\boldsymbol{\rho}(\zeta)]$ in Eq. (3.5) reflects the statistics of the medium, and, for instance, in the case of Gaussian fluctuations contains only a combination of correlation (structure) functions of the scattering potential $\tilde{\epsilon}(\mathbf{r},z,\omega)$. The remaining procedure is then to evaluate the path integral (3.5).

The term ''evaluation'' is applied to the path integral in the sense that the functional, i.e., infinite-dimensional, integral is reduced to some representation (computational algorithm) containing only finite-dimensional, conventional integrals. As is well known, the path integral may be evaluated exactly only in a very limited number of cases where, as a rule, the solution may be obtained by application of other methods [23]. This is true, first of all, for the Gaussian-type path integrals, i.e., the functional integrals with a quadratic-form Lagrangian. Such integrals may be handled by direct application of their discretized forms or orthogonal path expansions. Another way is to use the method of stationary phase which leads to exact results for quadratic Lagrangians, because the Taylor expansion of the corresponding actions terminates after its second derivative. However, in practice, especially in the theory of wave propagation in random media, the path integrals are of a non-Gaussian type, and the solution can be obtained in an approximate form only [22,24].

In the present study we will resort to a cumulant technique [25]. The general idea of the cumulant path integral evaluation is based on the notion of an expectation value introduced for an arbitrary real-valued functional $U[\boldsymbol{\rho}(\zeta)]$,

$$\langle U[\boldsymbol{\rho}(\zeta)]\rangle_X=\frac{\int D\boldsymbol{\rho}(\zeta)\exp\{-X[\boldsymbol{\rho}(\zeta)]\}U[\boldsymbol{\rho}(\zeta)]}{\int D\boldsymbol{\rho}(\zeta)\exp\{-X[\boldsymbol{\rho}(\zeta)]\}}. \quad (3.6)$$

Obviously, the definition (3.6) satisfies the necessary normalization condition $\langle 1\rangle_X=1$ (see Ref. [25]).

In order to calculate the value of Γ defined by Eq. (3.5), we first choose a trial action X_t, which on the one hand is close to the actual action X, and on the other is solvable, i.e., the value of the integral

$$\Gamma_t=\int D\boldsymbol{\rho}(\zeta)\exp\{-X_t[\boldsymbol{\rho}(\zeta)]\} \quad (3.7)$$

can be obtained analytically. In the second stage, by using the definition (3.6), we arrive at the expression

$$\Gamma=\Gamma_t\langle\exp[-(X-X_t)]\rangle_{X_t}, \quad (3.8)$$

in which the expectation value of the exponent may be replaced formally by the exponent of the series over corresponding cumulants κ_n:

$$\Gamma=\Gamma_t\exp(-\chi),\quad \chi=\sum_{n=1}^{\infty}\frac{\kappa_n}{n!}. \quad (3.9)$$

The cumulants κ_n are expressed through the moments

$$\mu_n=\langle(X-X_t)^n\rangle_{X_t},\quad n=1,2,\ldots, \quad (3.10)$$

by the usual nonlinear relations. A reasonable approximation can be obtained by terminating the series at the second order, which is allowed if a Gaussian nature is assumed in the stochastic behavior of the perturbation [25]. Under some conditions, the higher cumulants may be neglected, and the series can be approximated by the first cumulant only: $\chi\approx\kappa_1$. Obviously, some independent procedure aimed at the verification of the result based on the first cumulant is desirable. As such a procedure we use the evaluation of temporal moments of the pulse and compare the relative contributions of the first two cumulants to the pulse width (see Sec. VI for details).

IV. CALCULATIONS

We consider two waves with angular frequencies $\omega_1=\omega+\Omega/2$ and $\omega_2=\omega-\Omega/2$. Our aim is to evaluate a two-frequency correlator of the form (2.6), but generalized so that the observation points for the two waves are different. Specifically, we assume that the wave with frequency ω_n ($n=1,2$) is radiated by a point source located at $\mathbf{r}_{0n}$ in the plane $z_0=0$, and measured at the point $\mathbf{r}_n$ in the observation plane $z=L$. To calculate the corresponding propagators entering the coherence function,

$$\Gamma(\omega,\Omega)=\langle g_{\omega_1}(\mathbf{r}_1,z|\mathbf{r}_{01},z_0)g^*_{\omega_2}(\mathbf{r}_2,z|\mathbf{r}_{02},z_0)\rangle, \quad (4.1)$$

we use the velocity representation (3.4) in which the integration paths $\mathbf{v}_n(\zeta)$ are rescaled as

$$\mathbf{v}_n(\zeta)\to\alpha_n\mathbf{v}_n(\zeta), \quad (4.2)$$

where the coefficients α_n are given by

$$\alpha_n\equiv\alpha_n(\omega,\Omega)=\sqrt{k(\omega)/k(\omega_n)}, \quad (4.3)$$

and $k \equiv k(\omega)$ is the wave number corresponding to the "central" frequency ω. Also, we introduce the secondary coefficients

$$\alpha = \tfrac{1}{2}(\alpha_1 + \alpha_2), \quad \beta = \alpha_1 - \alpha_2, \quad \nu = \alpha\beta \tag{4.4}$$

and

$$\eta_1 = 1/\alpha_1^2, \quad \eta = 1/\alpha_1\alpha_2, \quad \eta_2 = 1/\alpha_2^2. \tag{4.5}$$

As we will see in the following, the value of $|\nu|$ is a naturally arising parameter that can serve as a measure of frequency separation. Note that ν is negative for normal dispersion and positive in the case of anomalous dispersion.

Now, using the identity

$$\delta(\alpha \mathbf{x}) = \alpha^{-m}\delta(\mathbf{x}), \quad \alpha > 0, \tag{4.6}$$

valid for any m-dimensional vector $\mathbf{x}$, we present the propagators g_{ω_n} in the form

$$\begin{aligned} g_{\omega_n}(\mathbf{r}_n, z|\mathbf{r}_{0n}, z_0) &= \alpha_n^{-m}\int D\mathbf{v}_n(\zeta)\,\delta\left(\mathbf{p}_{0n} - \mathbf{p}_n + \int_0^L d\zeta\, \mathbf{v}_n(\zeta)\right) \\ &\times \exp\left\{ i\frac{k}{2}\int_0^L d\zeta \left[\mathbf{v}_n^2(\zeta) \right.\right. \\ &\left.\left. + \alpha_n^{-2}\tilde{\epsilon}\left(\mathbf{r}_{0n} + \alpha_n \int_0^\zeta d\zeta\, \mathbf{v}_n(\zeta), \zeta, \omega_n \right)\right]\right\}, \end{aligned} \tag{4.7}$$

where the "normalized" coordinates $\mathbf{p}_{0n}$ and $\mathbf{p}_n$ are given by

$$\mathbf{p}_{0n} = \mathbf{r}_{0n}/\alpha_n, \quad \mathbf{p}_n = \mathbf{r}_n/\alpha_n. \tag{4.8}$$

Then, substituting the propagators g_{ω_n} into Eq. (4.1), introducing the Wigner-type paths

$$\mathbf{w}(\zeta) = \tfrac{1}{2}[\mathbf{v}_1(\zeta) + \mathbf{v}_2(\zeta)], \quad \mathbf{v}(\zeta) = \mathbf{v}_1(\zeta) - \mathbf{v}_2(\zeta), \tag{4.9}$$

and performing ensemble averaging, we arrive at the expression

$$\begin{aligned} \Gamma(\omega,\Omega) = \eta^m \Xi(\omega,\Omega) &\int D\mathbf{w}(\zeta) \int D\mathbf{v}(\zeta) \\ &\times \delta\left(\mathbf{P}_0 - \mathbf{P} + \int_0^L d\zeta\, \mathbf{w}(\zeta)\right) \\ &\times \delta\left(\mathbf{p}_0 - \mathbf{p} + \int_0^L d\zeta\, \mathbf{v}(\zeta)\right) \\ &\times \exp\left[ik\int_0^L d\zeta\, \mathbf{w}(\zeta)\cdot\mathbf{v}(\zeta)\right] \\ &\times \exp\{-X[\mathbf{w}(\zeta), \mathbf{v}(\zeta)]\}, \end{aligned} \tag{4.10}$$

in which we have defined the sum and difference coordinates

$$\mathbf{P}_0 = \tfrac{1}{2}(\mathbf{p}_{01} + \mathbf{p}_{02}), \quad \mathbf{p}_0 = \mathbf{p}_{01} - \mathbf{p}_{02}, \tag{4.11a}$$

$$\mathbf{P} = \tfrac{1}{2}(\mathbf{p}_1 + \mathbf{p}_2), \quad \mathbf{p} = \mathbf{p}_1 - \mathbf{p}_2. \tag{4.11b}$$

Analogous definitions are valid for the regular coordinate pairs $(\mathbf{R}_0, \mathbf{r}_0)$ and $(\mathbf{R}, \mathbf{r})$, introduced, respectively, in the source and observation planes. The functionals Ξ and X for the Markovian media with Gaussian statistics are expressed through the transverse correlation function $A_\epsilon(\mathbf{r}, \zeta, \omega, \Omega)$ and corresponding structure function $H_\epsilon(\mathbf{r}, \zeta, \omega, \Omega)$ (see Ref. [26] for definitions), and are given by

$$\begin{aligned} \Xi(\omega,\Omega) = \exp\Bigg(-\frac{k^2}{4}\int_0^L d\zeta [\, &\eta_1^2 A_\epsilon(\mathbf{0}, \zeta, \omega_1, 0) \\ &- 2\eta^2 A_\epsilon(\mathbf{0}, \zeta, \omega, \Omega) + \eta_2^2 A_\epsilon(\mathbf{0}, \zeta, \omega_2, 0)] \Bigg), \end{aligned} \tag{4.12}$$

$$X[\mathbf{w}(\zeta), \mathbf{v}(\zeta)] = \eta^2 \frac{k^2}{4}\int_0^L d\zeta\, H_\epsilon(\mathbf{r}(\zeta), \zeta, \omega, \Omega), \tag{4.13}$$

where

$$\mathbf{r}(\zeta) \equiv \mathbf{r}[\mathbf{w}(\zeta), \mathbf{v}(\zeta), \zeta] = \mathbf{r}_0 + \alpha\int_0^\zeta d\zeta\, \mathbf{v}(\zeta) + \beta\int_0^\zeta d\zeta\, \mathbf{w}(\zeta). \tag{4.14}$$

Since the double path integral entering Eq. (4.10) can be considered as a version of the general representation (3.5), the two-step procedure described in Sec. III can now be applied to its evaluation. In the first step, we construct the trial action $X_t[\mathbf{v}(\zeta)]$ by replacing $\mathbf{r}(\zeta)$ in the functional (4.14) with

$$\mathbf{r}_t(\zeta) \equiv \mathbf{r}_t[\mathbf{v}(\zeta), \zeta] = \mathbf{r}_0 + \alpha\int_0^\zeta d\zeta\, \mathbf{v}(\zeta). \tag{4.15}$$

This choice is inspired by the fact that for the degenerate case of coinciding frequencies we have $\beta = 0$, the functional $\mathbf{r}(\zeta)$ does not depend on $\mathbf{w}(\zeta)$, and the corresponding path integral is solvable [22]. For different but rather close frequencies, the functional $\mathbf{r}(\zeta)$ does depend on the trajectories $\mathbf{w}(\zeta)$ but their contribution is of much less significance as compared with that of $\mathbf{v}(\zeta)$.

Now, in Eq. (4.10), we expand the δ function containing $\mathbf{w}(\zeta)$ as

$$\begin{aligned} \delta\left(\mathbf{P}_0 - \mathbf{P} + \int_0^L d\zeta\, \mathbf{w}(\zeta)\right) &= (2\pi)^{-m}\int d\mathbf{u}\, \exp[i\mathbf{u}\cdot(\mathbf{P} - \mathbf{P}_0)] \\ &\times \exp\left(-i\mathbf{u}\cdot\int_0^L d\zeta\, \mathbf{w}(\zeta)\right), \end{aligned} \tag{4.16}$$

and the trial coherence function becomes

$$\Gamma_t(\omega,\Omega)=(2\pi)^{-m}\eta^m\Xi(\omega,\Omega)\int d\mathbf{u}\exp[i\mathbf{u}\cdot(\mathbf{P}-\mathbf{P}_0)]\times\int D\mathbf{v}(\zeta)\,\delta\left(\mathbf{p}_0-\mathbf{p}+\int_0^L d\zeta\,\mathbf{v}(\zeta)\right)\times\exp\{-X_t[\mathbf{v}(\zeta)]\}\int D\mathbf{w}(\zeta)\times\exp\left(ik\int_0^L d\zeta\,\mathbf{w}(\zeta)\cdot[\mathbf{v}(\zeta)-\mathbf{u}/k]\right). \tag{4.17}$$

By using the definition of the δ functional [22],

$$\delta[\mathbf{v}(\zeta)]=\int D\mathbf{w}(\zeta)\exp\left(ik\int_0^L d\zeta\,\mathbf{w}(\zeta)\cdot\mathbf{v}(\zeta)\right), \tag{4.18}$$

we integrate in Eq. (4.17) over $\mathbf{w}(\zeta)$, which leads to

$$\Gamma_t(\omega,\Omega)=(2\pi)^{-m}\eta^m\Xi(\omega,\Omega)\int d\mathbf{u}\exp[i\mathbf{u}\cdot(\mathbf{P}-\mathbf{P}_0)]\times\int D\mathbf{v}(\zeta)\,\delta\left(\mathbf{p}_0-\mathbf{p}+\int_0^L d\zeta\,\mathbf{v}(\zeta)\right)\times\delta[\mathbf{v}(\zeta)-\mathbf{u}/k]\exp\{-X_t[\mathbf{v}(\zeta)]\}. \tag{4.19}$$

By calculating also the integrals over $\mathbf{v}(\zeta)$ and $\mathbf{u}$, in the latter case using Eq. (4.6), we arrive at

$$\Gamma_t(\omega,\Omega)=(k/2\pi L)^m\eta^m\Xi(\omega,\Omega)\times\exp\left(i\frac{k}{L}(\mathbf{P}-\mathbf{P}_0)\cdot(\mathbf{p}-\mathbf{p}_0)\right)\times\exp\left(-\eta^2\frac{k^2}{4}\int_0^L d\zeta\,H_\epsilon(\mathbf{q}(\zeta),\zeta,\omega,\Omega)\right), \tag{4.20}$$

where

$$\mathbf{q}(\zeta)=\mathbf{r}_0+\alpha(\zeta/L)(\mathbf{p}-\mathbf{p}_0). \tag{4.21}$$

Note that for the degenerate one-frequency case we obtain the exact result [22] since in this case the trial action used coincides with the actual action functional.

At the second stage of the integration procedure, we construct a cumulant representation of the form

$$\Gamma(\omega,\Omega)=\Gamma_t(\omega,\Omega)\exp[-\chi(\omega,\Omega)], \tag{4.22}$$

where $\chi(\omega,\Omega)$ is a standard cumulant series [cf. Eqs. (3.9)]. The first cumulant reads

$$\kappa_1(\omega,\Omega)\equiv\mu_1(\omega,\Omega)=\eta^{m+2}\Xi(\omega,\Omega)\Gamma_t^{-1}(\omega,\Omega)\frac{k^2}{4}\int_0^L dz\int D\mathbf{w}(\zeta)\int D\mathbf{v}(\zeta)\,\delta\left(\mathbf{P}_0-\mathbf{P}+\int_0^L d\zeta\,\mathbf{w}(\zeta)\right)\times\delta\left(\mathbf{p}_0-\mathbf{p}+\int_0^L d\zeta\,\mathbf{v}(\zeta)\right)\exp\left[ik\int_0^L d\zeta\,\mathbf{w}(\zeta)\cdot\mathbf{v}(\zeta)\right]\exp\{-X_t[\mathbf{v}(\zeta)]\}\times\{H_\epsilon(\mathbf{r}[\mathbf{w}(\zeta),\mathbf{v}(\zeta),z],z,\omega,\Omega)-H_\epsilon(\mathbf{r}_t[\mathbf{v}(\zeta),z],z,\omega,\Omega)\}. \tag{4.23}$$

In order to perform the path integration in Eq. (4.23), we replace the transverse structure functions $H_\epsilon(\mathbf{r})$ by their spectral expansions,

$$H_\epsilon(\mathbf{r},z,\omega,\Omega)=4\pi\int d\mathbf{s}[1-\exp(i\mathbf{r}\cdot\mathbf{s})]\Phi_\epsilon(\mathbf{s},z,\omega,\Omega). \tag{4.24}$$

This allows us to present the cumulant κ_1 as a convolution

$$\kappa_1(\omega,\Omega)=\pi k^2\eta^2\int_0^L dz\int d\mathbf{s}\,f_1(\mathbf{s},z,\omega,\Omega)\Phi_\epsilon(\mathbf{s},z,\omega,\Omega) \tag{4.25}$$

of the power spectrum $\Phi_\epsilon(\mathbf{s},z,\omega,\Omega)$ with a filtering function $f_1(\mathbf{s},z,\omega,\Omega)$ given by

$$f_1(\mathbf{s},z,\omega,\Omega)=\eta^m\Xi(\omega,\Omega)\Gamma_t^{-1}(\omega,\Omega)\int D\mathbf{w}(\zeta)\int D\mathbf{v}(\zeta)\,\delta\left(\mathbf{P}_0-\mathbf{P}+\int_0^L d\zeta\,\mathbf{w}(\zeta)\right)\delta\left(\mathbf{p}_0-\mathbf{p}+\int_0^L d\zeta\,\mathbf{v}(\zeta)\right)\times\exp\left[ik\int_0^L d\zeta\,\mathbf{w}(\zeta)\cdot\mathbf{v}(\zeta)\right]\exp\{-X_t[\mathbf{v}(\zeta)]\}(\exp\{i\mathbf{s}\cdot\mathbf{r}_t[\mathbf{v}(\zeta),z]\}-\exp\{i\mathbf{s}\cdot\mathbf{r}[\mathbf{w}(\zeta),\mathbf{v}(\zeta),z]\}). \tag{4.26}$$

The simple trick now is to represent the last exponent in Eq. (4.26) as

$$\exp\{i\mathbf{s}\cdot\mathbf{r}[\mathbf{w}(\zeta),\mathbf{v}(\zeta),z]\} = \exp\left[ik\int_0^L d\zeta\,\mathbf{w}(\zeta)\cdot\boldsymbol{v}(\zeta)\right]\exp\{i\mathbf{s}\cdot\mathbf{r}_t[\mathbf{v}(\zeta),z]\}, \tag{4.27}$$

where

$$\boldsymbol{v}(\zeta)=\beta\vartheta(z-\zeta)\mathbf{s}/k, \tag{4.28}$$

$\vartheta(\zeta)$ is the Heaviside step function. The path integral then reduces to the form of Eq. (4.17), and the same integration procedure as that applied to the latter equation gives

$$f_1(\mathbf{s},z,\omega,\Omega)=\exp[i\mathbf{s}\cdot\mathbf{q}(z)]\{1-\exp[i\mathbf{s}\cdot\mathbf{Q}(z)]F_1\}, \tag{4.29a}$$

$$F_1=\exp\left(-i\sigma(z)s^2-\eta^2\frac{k^2}{4}\int_0^L d\zeta[H_\epsilon(\mathbf{q}(\zeta)-\sigma(z,\zeta)\mathbf{s},\zeta,\omega,\Omega)-H_\epsilon(\mathbf{q}(\zeta),\zeta,\omega,\Omega)]\right), \tag{4.29b}$$

where

$$\mathbf{Q}(z)=\beta(z/L)(\mathbf{P}-\mathbf{P}_0) \tag{4.30}$$

and

$$\sigma(z)=\nu z(1-z/L)/k, \tag{4.31a}$$

$$\sigma(z_1,z_2)=\nu[\min(z_1,z_2)-z_1z_2/L]/k. \tag{4.31b}$$

Analogous calculations performed with the second cumulant yield

$$\kappa_2(\omega,\Omega)\equiv\mu_2(\omega,\Omega)-\mu_1^2(\omega,\Omega) = \pi^2k^4\eta^4\int_0^L dz_1\int_0^L dz_2\int d\mathbf{s}_1 \times\int d\mathbf{s}_2[f_2(\mathbf{s}_1,\mathbf{s}_2,z_1,z_2,\omega,\Omega) - f_1(\mathbf{s}_1,z_1,\omega,\Omega)f_1(\mathbf{s}_2,z_2,\omega,\Omega)] \times\Phi_\epsilon(\mathbf{s}_1,z_1,\omega,\Omega)\Phi_\epsilon(\mathbf{s}_2,z_2,\omega,\Omega), \tag{4.32}$$

where the filtering function f_2 is given by

$$f_2(\mathbf{s}_1,\mathbf{s}_2,z_1,z_2,\omega,\Omega) =\exp[i\mathbf{s}_1\cdot\mathbf{q}(z_1)+i\mathbf{s}_2\cdot\mathbf{q}(z_2)] \times\{1-\exp[i\mathbf{s}_1\cdot\mathbf{Q}(z_1)]F_{11}-\exp[i\mathbf{s}_2\cdot\mathbf{Q}(z_2)]F_{22} +\exp[i\mathbf{s}_1\cdot\mathbf{Q}(z_1)+i\mathbf{s}_2\cdot\mathbf{Q}(z_2)]F_{12}\}. \tag{4.33a}$$

Here the factors F_{11} and F_{22} are defined as

$$F_{nn}=\exp\left\{-i\sigma(z_n)s_n^2-i\sigma(z_1,z_2)\mathbf{s}_1\cdot\mathbf{s}_2 -\eta^2\frac{k^2}{4}\int_0^L d\zeta[H_\epsilon(\mathbf{q}(\zeta)-\sigma(z_n,\zeta)\mathbf{s}_n,\zeta,\omega,\Omega) -H_\epsilon(\mathbf{q}(\zeta),\zeta,\omega,\Omega)]\right\}, \tag{4.33b}$$

and the factor F_{12} has the form

$$F_{12}=\exp\left(-i\sigma(z_1)s_1^2-2i\sigma(z_1,z_2)\mathbf{s}_1\cdot\mathbf{s}_2-i\sigma(z_2)s_2^2 -\eta^2\frac{k^2}{4}\int_0^L d\zeta[H_\epsilon(\mathbf{q}(\zeta)-\sigma(z_1,\zeta)\mathbf{s}_1 -\sigma(z_2,\zeta)\mathbf{s}_2,\zeta,\omega,\Omega)-H_\epsilon(\mathbf{q}(\zeta),\zeta,\omega,\Omega)]\right). \tag{4.33c}$$

The cumulant representation (4.22), together with the expressions for $\Gamma_t(\omega,\Omega)$ and $\chi(\omega,\ \Omega)$, is the main result of this work. In contrast to the formulas available in the literature, the solution obtained here is not limited by the strength of disorder and applies equally well to both dispersive and nondispersive media, with arbitrary spectra of inhomogeneities. It is important that the dependence of even the first cumulant $\kappa_1(\omega,\Omega)$ on the medium fluctuations is highly nonlinear, and has nothing in common with the quadratic approximation of the transverse structure function.

V. ANALYSIS

As will be shown in the next section, the contribution of the second cumulant to the value of $\chi(\omega,\Omega)$ is rather small as compared with that of the first one. Therefore, in the following we will restrict ourselves to the approximation $\chi(\omega,\Omega)\approx\kappa_1(\omega,\Omega)$, although more exact calculations including the second or even higher cumulants can be performed. To simplify the analysis, we will focus on the case of zero spatial separation in both source and observation planes, $\mathbf{r}_{0n}=\mathbf{0}$ and $\mathbf{r}_n=\mathbf{0}$. Then, the trial coherence function becomes

$$\Gamma_t(\omega,\Omega)=(k/2\pi L)^m\eta^m\Xi(\omega,\Omega). \tag{5.1}$$

Being determined by the integral parameter $A_\epsilon(\mathbf{0})$, the function $\Gamma_t(\omega,\Omega)$ depends significantly on the behavior of the spectrum at small spatial frequencies. In the case of fractal media, $\Gamma_t(\omega,\Omega)$ accounts for the optical path length variations and, as a result, contributes entirely to the fluctuations in the arrival time of the pulse. The second factor entering

Eq. (4.22), $\exp[-\chi(\omega,\Omega)]$, is much less sensitive to the low-frequency behavior of the spectrum, and determines the distortions of the transient signal due to its scattering in a random medium. Therefore, we will concentrate on the analysis of the latter factor, setting hereafter $\Gamma_t(\omega,\Omega)=1$. Thus, the real part of the cumulant series defines the absolute value of the coherence function, $\mathrm{Re}\,\chi\equiv\ln|\Gamma|$, while the imaginary part corresponds to its phase, $\mathrm{Im}\,\chi\equiv\arg\Gamma$.

Also, we will consider a random medium that is characterized by isotropic scattering in the transverse plane and is statistically homogeneous along the longitudinal coordinate. In this case Eqs. (4.25) and (4.29) are transformed to

$$\chi(\omega,\Omega)=2\pi^2k^2\eta^2\int_0^L dz\int_0^\infty ds\, s f(s,z,\omega,\Omega)\Phi_\epsilon(s,\omega,\Omega), \tag{5.2}$$

$$f(s,z,\omega,\Omega)=1-\exp\left(-i\sigma(z)s^2 - \eta^2\frac{k^2}{4}\int_0^L d\zeta\, H_\epsilon(\sigma(z,\zeta)s,\omega,\Omega)\right). \tag{5.3}$$

The physical meaning of Eqs. (5.2) and (5.3) is fairly transparent: the frequency separation of two different waves is converted into a spatial separation of the corresponding rays due to the effect of diffraction on the inhomogeneities of the medium. A very important feature is the dependence of this spatial separation on the spatial frequency of the disorder. The higher the frequency is (the smaller the spatial scale) the larger the separation of the diffracted rays. Also, the form of the filtering function (5.3) suggests a simple classification of the propagation regimes. As can be seen, the critical parameter here is the distance of propagation L. For relatively *short distances* (in the regime of rather *weak scattering*) the first term in the exponent dominates, and the correlation of two signals with different frequencies is determined by the transverse size of the first Fresnel zone, $\sqrt{L/k}$, and the frequency separation ν. It is worth noting that for a time-harmonic wave propagating in a turbulent medium, both weak and strong intensity fluctuations can be observed in this regime. For *long distances* (the regime of *strong scattering*) the first term in the exponent is smaller than the second one, and the spectral filtering depends essentially on the behavior of the structure function H_ϵ. In this regime, the intensity fluctuations of a time-harmonic wave are well saturated. A complementary point of view for the classification of propagation regimes will be discussed in Sec. VI.

Although the integration in Eqs. (5.2) and (5.3) may be carried out numerically, in some practically important situations the filtering function can be simplified further. In fact, although the integration over s in Eq. (5.2) is performed formally up to infinity, for any real random medium there exists an upper cutoff frequency s_m above which the spectrum $\Phi_\epsilon(s)$ quickly decays. The spatial frequency s_m is determined by the lowest scale of the disorder. In a turbulent medium, for example, this is the so-called inner scale l_0, which separates inertial and viscous intervals of the turbulent spectrum [27]. If the condition

$$|\nu|\ll kl_0^2/L \tag{5.4}$$

holds, for all frequencies s contributing significantly to the value of the integral in Eq. (5.3), we have

$$|\sigma(z,\zeta)|s\ll l_0. \tag{5.5}$$

Physically, this means that the spatial separation of the diffracted rays for two waves with different frequencies is smaller than the inner scale of the medium. For such an argument, the exact structure function $H_\epsilon(r)$ can be replaced by the first term of its series expansion,

$$H_\epsilon(r)=\frac{1}{2!}H''_\epsilon(0)r^2+\frac{1}{4!}H^{(4)}_\epsilon(0)r^4+\cdots. \tag{5.6}$$

Note that odd terms in the latter equation are absent due to the symmetry of the function. As a result, we obtain

$$f(s,z,\omega,\Omega)=1-\exp[-i\sigma(z)s^2-\rho^2(z)s^2], \tag{5.7}$$

where the parameter $\rho(z)$ is defined as

$$\rho^2(z)=\frac{1}{48}\nu^2\eta^2H''_\epsilon(0,\omega,\Omega)Lz^2(5-4z/L-z^2/L^2), \tag{5.8}$$

and has the meaning of a frequency-dependent coherence radius.

To exemplify the result we will consider here a simple but rather general and practically interesting model, which is characterized by a power spectrum of the form

$$\Phi_\epsilon(s,\omega,\Omega)=C(p,\omega,\Omega)(s_0^2+s^2)^{-p/2}\exp(-s^2/s_m^2), \tag{5.9}$$

where $C(p,\omega,\Omega)$ is a parameter defining the strength of disorder, taking into account also the dispersive properties of the medium and including the normalization constant, and s_m and s_0 are inversely proportional to the inner, l_0, and outer, L_0, scales, respectively. In particular, the case $p=11/3$ corresponds to a Kolmogorov turbulent spectrum modified in the regions of both small and large scales, and the value of C is proportional to the structure parameter of the turbulence, C_ϵ^2. Apart from continuous media, the general form of the spectrum (5.9) is capable of modeling also the cumulative effect of rather large discrete scatterers, such as hydrometeors (rain, fog, etc.) in the troposphere, which scatter the light predominantly in the forward direction and may cause an essential delay time and broadening of rather short optical and millimeter-wave pulses [1].

Substituting Eqs. (5.7) and (5.9) into Eq. (5.2) leads to

$$\chi(\omega,\Omega)=2\pi^2C(p,\omega,\Omega)\,\eta^2k^2\times\int_0^L dz\, s_0^{2-p}[\Psi(1,2-p/2;s_0^2/s_m^2)-\Psi(1,2-p/2;s_0^2/s_m^2+i\sigma(z)s_0^2+\rho^2(z)s_0^2)], \tag{5.10}$$

where $\Psi(a,b;z)$ is the Tricomi hypergeometric function. A simple asymptotic analysis of this result shows that for fractal spectra, such as Kolmogorov turbulence, the effect of both inner and outer scales may be significant even when the inertial interval is rather wide. However, to complete the analytical evaluations we will not consider this effect, assuming that the value of p satisfies the condition $3<p<4$ and $s_0\to 0$, $s_m\to\infty$. This gives

$$\chi(\omega,\Omega)=4\pi^2(p-2)^{-1}\Gamma(2-p/2)\times C(p,\omega,\Omega)\,\eta^2k^2\int_0^L dz[i\sigma(z)+\rho^2(z)]^{p/2-1}, \tag{5.11}$$

where $\Gamma(z)$ is the gamma function. In the regime of weak scattering we obtain

$$\chi(\omega,\Omega)=p[\sin(p\pi/4)\Gamma(p)]^{-1}\Gamma^2(p/2)\times\beta_0^2(p,\omega,\Omega)\,\eta^2(i\nu)^{p/2-1}, \tag{5.12}$$

where the dimensionless parameter

$$\beta_0^2(p,\omega,\Omega)=4\pi^2[p(p-2)]^{-1}\sin(p\pi/4)\times\Gamma(2-p/2)C(p,\omega,\Omega)k^{3-p/2}L^{p/2} \tag{5.13}$$

is a normalized variance of intensity fluctuations evaluated in the framework of the Rytov approximation [27]. The natural scaling is provided by the value of $\nu^{p/2-1}$. In terms of this parameter, the magnitude of the coherence function exponentially decays, accompanied by a linear phase accumulation. For the Kolmogorov turbulence $p=11/3$ taken as a specific example, we have $-\ln\Gamma\sim\nu^{5/6}$. This scaling is in excellent agreement with the results obtained by solving the differential equation for the coherence function numerically (see Ref. [1]). As follows from Eq. (5.11), in the regime of strong scattering, which has not been covered in the previous studies, the coherence function becomes purely real and is characterized by the relation $-\ln\Gamma\sim\nu^{5/3}$.

When the inequality (5.4) is not satisfied, the behavior of the filtering function is not universal and depends essentially on the exact form of the power spectrum. A unified scaling, however, can be predicted for the temporal moments characterizing the evolution of pulsed waves in random media.

VI. APPLICATION TO PULSE PROPAGATION

Having at hand the expression for the mutual coherence function, and using Eq. (2.4), we can evaluate the mean shape of the transient signal propagating in a random medium. However, the final solution $\langle I(\mathbf{r},z,t)\rangle$ in some cases contains more information than is actually needed. If we are interested in finding only the integral parameters of the wave packet, such as the mean arrival time or pulse width, the evolution of transient waves can be described easily by temporal moments, a technique commonly used in quantum mechanics, and adopted in Ref. [14] for random propagation problems.

The temporal moments of the transient field are introduced as

$$t_n(\mathbf{r},z)=\int_{-\infty}^{\infty}dt\, t^n\langle I(\mathbf{r},z,t)\rangle. \tag{6.1}$$

We suppose that the intensity is normalized to satisfy the condition $t_0(\mathbf{r},z)=1$ imposed on the total energy of the pulse. The first moment $t_1(\mathbf{r},z)$ is associated with the mean arrival time,

$$\tau(\mathbf{r},z)=t_1(\mathbf{r},z)-t_1(\mathbf{r},0), \tag{6.2}$$

and the second moment $t_2(\mathbf{r},z)$ defines the pulse width,

$$w^2(\mathbf{r},z)=t_2(\mathbf{r},z)-t_1^2(\mathbf{r},z). \tag{6.3}$$

After some manipulations it can be found [14] that

$$t_n(\mathbf{r},z)=(-i)^n\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\,\frac{\partial^n}{\partial\Omega^n}\times[\Phi(z,\omega,\Omega)\Gamma(\mathbf{r},z,\omega,\Omega)]_{\Omega=0}. \tag{6.4}$$

According to Eq. (6.4), the temporal moments of the pulse are given by derivatives of the bilinear spectrum Φ and the two-frequency coherence function Γ, both calculated at $\Omega=0$. For rather narrowband signals, where the coherence function Γ and its derivatives do not change significantly within the effective frequency band, Eq. (6.4) may be simplified. In particular, the mean arrival time and the pulse width are given by

$$\tau(\mathbf{r},z)=\tau_0(\mathbf{r},z)+i\chi'(\mathbf{r},z,\omega,0), \tag{6.5}$$

$$w^2(\mathbf{r},z)=w_0^2(\mathbf{r},z)+\chi''(\mathbf{r},z,\omega,0), \tag{6.6}$$

where $\tau_0(\mathbf{r},z)$ and $w_0(\mathbf{r},z)$ are the mean arrival time and the pulse width for the wave propagating in a homogeneous medium. Analogously, any other (high-order) temporal moment can be calculated, with a corresponding increase of algebraic complexity being the only limitation. The main advantage of working with temporal moments is now clearly seen, since the evaluation of temporal moments may be performed analytically. Indeed, there is no need to calculate the function $\Gamma(\omega,\Omega)$ itself, and it is sufficient to find only its first derivatives taken at $\Omega=0$. To obtain the exact analytical expressions for the nth temporal moment we have to calculate the corresponding number of cumulants. In particular, performing appropriate calculations for a nondispersive medium by using the first cumulant, we have

$$\tau(\mathbf{r},z)=\tau_0(\mathbf{r},z)+\frac{1}{24}H_\epsilon''(0)L^2/c. \tag{6.7}$$

The contribution of the second cumulant to this temporal moment is exactly zero. To calculate the pulse width we have to account for the first two cumulants. The result is

$$w^2(\mathbf{r},z)=w_0^2(\mathbf{r},z)+\frac{1}{180}|H_\epsilon^{(4)}(0)|L^3/k^2c^2 + \frac{1}{480}[H_\epsilon''(0)]^2L^4/c^2. \tag{6.8}$$

Here, the first term, which is proportional to L^3, originates from the first cumulant and corresponds to the regime of weak scattering (or relatively short propagation distances). The second term, with L^4 behavior, corresponds to the regime of strong scattering (large distances). The coefficient 1/480 is composed of the term 7/2880 coming from the first cumulant, and the term $-1/2880$ related to the second cumulant. We can conclude, therefore, that for a value of $w(\mathbf{r},z)$ the relative error of the approximation based on the first cumulant is rather small even in the regime of strong scattering.

VII. SUMMARY

In this work, the temporal evolution of transient waves propagating in forward scattering random media has been studied. The analysis is performed in the frequency domain and is based on the solution for the two-frequency mutual coherence function. The result obtained for this function may serve as a good example of how the path integral technique applied to a random propagation problem can give a final expression in a simple form with well-controlled accuracy, despite the intermediate procedure being rather complicated. In contrast to the formulas available in the literature, the solution obtained is not limited by the strength of disorder and works equally well in both dispersive and nondispersive media. Generalization of the expression for the coherence function to finite-aperture wave beams is straightforward. The dependence of the spectrum Φ_ϵ on the longitudinal coordinate and anisotropic scattering in the transverse plane can be accounted for in the calculations for particular applications. In principle, the results can be generalized also to random media with a regular refraction profile, as is important, e.g., for high-frequency wave propagation in an inhomogeneous ionosphere.

Two different regimes of propagation have been found. For the regime of relatively weak scattering (or rather short propagation distances) the analytical expressions obtained are in excellent agreement with known numerical results. In this regime the description of the coherence function by only the first cumulant is very accurate. In many situations (as, e.g., for laser pulses of picosecond or femtosecond duration propagating in a turbulent atmosphere) the regime of strong scattering never sets in for real distances, and therefore the weak scattering approximation based on the first cumulant provides an essentially exact picture. At the same time, there are many applications (such as, e.g., optical beams transmitted through forward scattering particulate media) when the regime of strong scattering can occur. In this regime, which to our knowledge has not been covered earlier, the scaling differs essentially from that for the weak scattering, depending much more strongly on the frequency separation. It is shown that in random media with fractal-like correlations, the exact behavior of the spectrum at both small and large spatial frequencies is important.

Using the cumulant expansion we have considered also the temporal moments of a pulsed wave propagating in a random medium. It has been found that the temporal moments of the pulse are determined exactly by accounting for a corresponding number of the cumulants entering the expression for the coherence function. In particular, the average time delay of the pulse is determined by the first cumulant, and the pulse width is obtained by accounting for the first two cumulants. In the regime of strong scattering, the approximation based on the first cumulant overestimates to some extent the decay rate of the coherence function, and, as a result, predicts an increased broadening of the pulse. Better accuracy for the coherence function is achieved by accounting also for the second cumulant, which gives an exact value of the pulse width.

ACKNOWLEDGMENT

This work was supported, in part, by the Office of Naval Research, under Grant No. N00014-00-1-0672.

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